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THE TEMPERATURE AT THE PHASE BOUNDARY

DURING QUASI-STATIC MOTIONS

OF A TWO-PHASE THERMOELASTIC BAR

by

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1. Introduction

In this paper, a thermo-mechanical problem involving a finite bar is considered. It is assumed that the bar initially consists of two phases, one of which was obtained from the other by a martensitic phase transformation. It is also assumed that both phases of the bar have thermoelastic constitutive behavior. The two-phase bar is subject to different temperatures at each boundary and to a boundary traction. A quasi-static thermo-mechanical process is considered and the temperature, static equilibrium positions, and quasi-static motions of the phase boundary are studied as the material coefficients, temperature boundary conditions, and end loading are varied.

Martensitic phase transformations are diffusionless solid-solid phase transformations which have continuous displacements, with possible discontinuous strains, at their phase boundary [2], [5]. They are also characterized by the parent and product phases having different crystal structures, or at least different crystal orientations, and by the product phase having a shape deformation relative to the undeformed parent phase, which corresponds to an unstressed undeformed state of that product phase. The high-temperature phase of a material that can undergo such a transformation is usually referred to as the austenite phase of the material, and the low-temperature phase that can be created from the austenite by a martensitic phase transformation is usually referred to as the martensite phase of the material. Additionally, there are usually several variants of the martensite phase. These types of transformations can be induced thermally or, in certain temperature intervals, by the application of certain types of mechanical loadings. In the absence of any external mechanical loading, the temperature at which the martensitic phase transformation first occurs as the temperature is lowered is called the martensitic start temperature (M_s), and the temperature at which the "reverse" martensitic transformation to the austenite phase

first occurs as the temperature is raised is called the austenite start temperature (A_s). These temperatures are material-dependent, and in most materials, $M_s < A_s$.[1]

In the following sections, the kinematics and the continuum model that was developed in [3] and is used here are discussed. The field equations for the thermo-mechanical quasi-static process that is considered are then presented. The constitutive assumptions for each phase corresponding to the continuum model that is used are discussed. These constitutive equations are then linearized about the unstressed undeformed configurations of their respective phases. Next, a kinetic relation with an interval of driving traction corresponding to a zero phase boundary velocity is considered. The linearized forms of this kinetic relation about three different temperatures are discussed. One of these temperatures can be considered to correspond to the martensitic start temperature of the material, and one of the temperatures can be considered to correspond to the austenite start temperature of the material. The boundary value problem is then considered. Because the goal is to study the temperature and quasi-static motions of the interface and, for the problem under consideration, the displacements are not needed for this, only the temperature distribution will be completely solved for. The temperature at the interface is then studied as the parameters of the problem are varied. Quasi-static motions in the neighborhoods of the three temperatures that were considered for the linearized kinetic relation are then considered. These quasi-static motions are studied as the parameters of the problem are varied.

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[1] See [2] and [5] for more detailed and comprehensive discussions about martensitic phase transformations.

2. The Kinematics

A one-dimensional finite bar that initially consists of two phases is considered. It is assumed that the process under consideration occurs in a time interval $\Gamma = [t_0, t_1]$. Additionally, for the problem that is considered, a continuum model that was developed in [3] (see also [4]) is used.

Consider a stationary reference configuration R for the bar. Let x denote a point in R and let L be the length of the bar with respect to this reference configuration. Considering this, R can be expressed as $R = \{x\,/\,x \in [0, L]\}$. Let $\hat{y}(x, t)$ be the suitably smooth and invertible mapping which maps R into the deformed configuration of the bar at each $t \in \Gamma$, with $\hat{y}(x, t) = x + \hat{u}(x, t) \;\forall\; (x, t) \in R \times \Gamma$. The quantity $\hat{u}(x, t)$ represents the displacement of a particle of material at $y = \hat{y}(x, t)$ from the point $x \in R$ at time $t \in \Gamma$. In the following, the two phases of the bar will be referred to as phase 1 and phase 2. Let $x = s(t)$ be the reference position of the phase boundary separating phase 1 from phase 2 at time $t \in \Gamma$. It is assumed that particles of material with reference points in $R^- = \{x\,/\,x \in [0, s(t)]\}$ at time $t \in \Gamma$ are in phase 1, and it is assumed that particles of material with reference points in $R^+ = \{x\,/\,x \in [s(t), L]\}$ at time $t \in \Gamma$ are in phase 2. It is assumed that R^- coincides with an unstressed undeformed configuration of phase 1. We next assume that there exists a shape deformation of phase 2 with respect to R^+ that corresponds to an unstressed undeformed configuration of that phase. Let R_1^+ be the reference configuration coinciding with this shape deformation for all $t \in \Gamma$. Let x_1 denote a point in R_1^+, and let $\tilde{x}_1(x, t)$ be the suitably smooth and invertible mapping which maps R^+ into R_1^+ at each $t \in \Gamma$, with $x_1 = \tilde{x}_1(x, t) \;\forall\; x \in R^+$ at each $t \in \Gamma$. For the following problem, it is assumed that $\tilde{x}_1$ is given by

$$\tilde{x}_1(x,t) = x + \int_{s(t)}^{x} \tilde{\gamma}(\theta(\xi))d\xi , \qquad (2.1)$$

where $\theta(x)$ is the temperature field of the material with respect to the coordinates of R. The displacement gradient (transformation strain) corresponding to this choice of $\tilde{x}_1$ is $\tilde{\gamma}(\theta(x))$, the Jacobian is $\tilde{J} = 1 + \tilde{\gamma}$, and $R_1^+ = \{x_1/\ x_1 \in [\tilde{x}_1(s(t),t), \tilde{x}_1(L,t)]\} = \{x_1/\ x_1 \in [s(t), L_1(t)]\}$, where $L_1(t) = \tilde{x}_1(L,t)$. It is also required that $\tilde{\gamma} > -1$ so that reflections are excluded from (2.1). The mapping which maps R_1^+ into the deformed configuration of phase 2 at each $t \in \Gamma$ is represented by $\hat{y}_1(x_1,t)$, with $\hat{y}_1(x_1,t) = x_1 + \hat{u}_1(x_1,t) \ \forall \ x_1 \in R_1^+$ at each $t \in \Gamma$.

3. The Continuum Model

In the continuum model that is used here, the constitutive equations for each phase are defined with respect to different reference configurations. More specifically, the constitutive equations for phase 1 are defined with respect to R^-, and the constitutive equations for phase 2 are defined with respect to R_1^+. Additionally, the field equations for phase 1 are expressed with respect to R^-, and the field equations for phase 2 are expressed with respect to R_1^+ (see [3], [4]). The main advantage of using this continuum model for the problem under consideration is that the field equations are in forms that permit direct linearization. This is the case since the displacements for each phase are measured from a reference configuration coinciding with an unstressed undeformed configuration of that phase, and consequently, for the appropriate boundary and initial conditions, the displacement gradients can be considered infinitesimal.

4. The Field Equations and Jump Conditions

It is assumed that the process under consideration is thermo-mechanical with no body forces and no heat sources present. It is further assumed that the process occurs in a temperature interval $\Omega = [\theta_0', \theta_1']$ containing the temperatures M_s and A_s. In the following, a quasi-static process is considered. In particular, a process is considered where time is treated as just a parameter in all quantities except for $s(t)$.[2] The dependence of $s(t)$ on time is such that $\left(\frac{ds(t)}{dt}\right)^n$ $(n > 1)$ and $\ddot{s}(t) = \frac{\partial^2 s(t)}{\partial t^2}$ are assumed to be negligible for all $t \in \Gamma$, and are therefore taken to be equal to zero. For this type of process, the only time derivative terms in the boundary value problem that are not identically equal to zero are those that are linear in $\dot{s}(t) = \frac{ds(t)}{dt}$. Such a process corresponds to a process where $|\dot{s}(t)| \ll 1$ and the acceleration of the phase boundary is negligible for all $t \in \Gamma$.

The general field equations using the type of continuum model described in previous section were derived and discussed in [3]. For the type of process under consideration, the balance of linear momentum, the balance of energy (the first law of thermodynamics), and the entropy inequality (the second law of thermodynamics) for phase 1 are

$$\frac{d\sigma}{dx} = 0,$$

$$\frac{dq}{dx} = 0, \tag{4.1}$$

$$\frac{d}{dx}\left(\frac{q}{\theta}\right) \leq 0,$$

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[2] Because of this, in the following, the dependence of all quantities, except for $s(t)$ and $\tilde{z}_1(x,t)$, on time will not be displayed in the notation. Also, strictly speaking, such a process is not a true quasi-static process since time is not just a parameter in all variables of the problem, and therefore the set of all solutions as time is varied does not consist of only static equilibrium solutions.

respectively, $\forall x \in (0, s(t))$ at each $t \in \Gamma$, where $\sigma(x)$ is the nominal stress with respect to R^- for phase 1, $q(x)$ is the nominal heat flux with respect to R^- for phase 1, and, as mentioned previously, $\theta(x)$ is the temperature field of the material with respect to the coordinates of R.

Similarly, for the type of process under consideration, the balance of linear momentum, the balance of energy, and the entropy inequality for phase 2 are

$$\frac{d\sigma_1}{dx_1} = 0,$$

$$\frac{dq_1}{dx_1} = 0, \tag{4.2}$$

$$\frac{d}{dx_1}\left(\frac{q_1}{\theta_1}\right) \leq 0,$$

respectively, $\forall x_1 \in (s(t), L_1(t))$ at each $t \in \Gamma$, where $\sigma_1(x_1)$ is the stress with respect to R_1^+ for phase 2, $q_1(x_1)$ is the heat flux with respect to R_1^+ for phase 2, and $\theta_1(x_1)$ is the temperature field of phase 2 with respect to the coordinates of R_1^+.

In addition to continuity of displacements at the phase boundary, we also require that the temperature be continuous. At $x = s(t)$, these conditions can be expressed as

$$u_1^+(s(t)) = u^-(s(t)),$$
$$\theta_1^+(s(t)) = \theta^-(s(t)), \tag{4.3}$$

respectively, at each $t \in \Gamma$,[3] where the "+" and "−" superscripts denote the limiting

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[3] Note that the original form of the continuity of displacement condition $y_1^+ = y^-$ reduces to (4.3)$_1$ for the $\bar{x}_1$ given by (2.1).

values of their respective quantities from the "+" and "−" sides of $x = s(t)$, respectively, at time $t \in \Gamma$.

As mentioned in the Introduction, the strains at the phase boundary separating two phases involved in a martensitic phase transformation may be discontinuous. Considering this, we require that the first and second derivatives of $\hat{y}$ be piecewise continuous on R, with possible discontinuities occurring only at $x = s(t)$. In addition to this, we require that the stress, entropy, and heat flux be piecewise continuous, with possible discontinuities occurring only at $x = s(t)$. The jump conditions at $x = s(t)$ that are equivalent to the balance of linear momentum, the balance of energy, and the entropy inequality are given by

$$\sigma_1^+ - \sigma^- = 0,$$

$$\left\{ \theta\left(\tilde{J}\eta_1^+ - \eta^- \right) + f \right\}\dot{s} + \left(q_1^+ - q^- \right) = 0, \tag{4.4}$$

$$f\dot{s} \geq 0,$$

respectively, at each $t \in \Gamma$, where $\eta(x)$ is the entropy of phase 1 with respect to R^-, $\eta_1(x_1)$ is the entropy of phase 2 with respect to R_1^+, and

$$f = \tilde{J}\psi_1^+ - \psi^- - \sigma^-\left(\tilde{\gamma} + \gamma_1^+\tilde{\gamma} + \gamma_1^+ - \gamma^- \right) \tag{4.5}$$

is the driving traction acting at the interface for the quasi-static process under consideration.[4] We also have at the phase boundary a kinetic relation [1] relating $\dot{s}$ to at least f. This kinetic relation is a constitutive relation and will be discussed in Section 6.

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[4] See [1] for the forms of the field equations and jump conditions that were presented in this section that correspond to a continuum model that uses a single elastic potential and only one reference configuration.

We must also specify the boundary conditions at $x = 0$ and $x_1 = L_1(t)$. The boundary conditions for Equations $(4.1)_1$ and $(4.2)_1$ are

$$u(0) = 0, \qquad \sigma_1(L_1(t)) = \sigma_0, \tag{4.6}$$

respectively, and the boundary conditions for the temperatures $\theta(x)$ and $\theta_1(x_1)$ are

$$\theta(0) = \theta^*, \qquad \theta_1(L_1(t)) = \theta_1^*, \tag{4.7}$$

respectively.

5. The Constitutive Assumptions

It is assumed that both phase 1 and phase 2 are homogeneous thermoelastic phases. In particular, for phase 1 it is assumed that there exists a Helmholtz free energy potential

$$\psi = \psi(\gamma, \theta), \tag{5.1}$$

such that

$$\sigma = \frac{\partial \psi}{\partial \gamma}, \tag{5.2}$$

$$\eta = -\frac{\partial \psi}{\partial \theta},$$

where $\gamma = \frac{du}{dx}$, and it is assumed that the heat flux response function for phase 1 has the form

$$q = \hat{q}\left(\gamma, \theta, \frac{d\theta}{dx}\right). \tag{5.3}$$

Similarly, for phase 2 it is assumed that there exists a Helmholtz free energy potential

$$\psi_1 = \psi_1(\gamma_1, \theta_1), \tag{5.4}$$

such that

$$\sigma_1 = \frac{\partial \psi_1}{\partial \gamma_1},$$

$$\eta_1 = -\frac{\partial \psi_1}{\partial \theta_1},$$
(5.5)

where $\gamma_1 = \frac{du_1}{dx_1}$, and it is assumed that the heat flux response function for phase 2 has the form[5]

$$q_1 = \hat{q}_1\left(\gamma_1, \theta_1, \frac{d\theta_1}{dx_1}\right).$$
(5.6)

We note that all of the quantities for phase 1 given by (5.1)-(5.3) are defined with respect to the reference configuration R^-, and all of the quantities for phase 2 given by (5.4)-(5.6) are defined with respect to the reference configuration R_1^+.

We next assume that the thermo-mechanical process under consideration takes place in a neighborhood of a temperature $\theta_0 \in \Omega$. More precisely, letting $\hat{\theta} = (\theta - \theta_0)$ and $\hat{\theta}_1 = (\theta_1 - \theta_0)$, we assume that $|\hat{\theta}| \ll 1 \; \forall x \in [0, s(t)]$ and $|\hat{\theta}_1| \ll 1$ $\forall x_1 \in [s(t), L_1(t)]$, at each $t \in \Gamma$. Additionally, we assume that for a given θ_0, the dependence of $\tilde{\gamma}$ on $\hat{\theta}$ is at least second-order; i.e. we assume

$$\tilde{\gamma} = \gamma_0 + O(\hat{\theta}^2),$$
(5.7)

where $\gamma_0 = \tilde{\gamma}(\theta_0) > -1$. For this assumption, the first-order approximation of $\tilde{x}_1$ is

$$\tilde{x}_1(x, t) = x + \gamma_0(x - s(t)),$$
(5.8)

and the first-order approximation of $L_1(t)$ is $L_1(t) = L + \gamma_0(L - s(t))$. We assume that phase 1 is unstressed at $(\gamma, \theta) = (0, \theta_0)$, and phase 2 is unstressed

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[5] The constitutive equations corresponding to a thermoelastic material defined with respect to a single fixed reference configuration are well known and can be found in, e.g., [1].

at $(\gamma_1, \theta_1) = (0, \theta_0)$. We also assume that $|\gamma| \ll 1 \; \forall \, x \in [0, s(t))$ and $|\gamma_1| \ll 1$ $\forall \, x_1 \in (s(t), L_1(t)]$, at each $t \in \Gamma$. Considering this, ψ for phase 1 can be written as

$$\psi = \psi^* - \eta^* \hat{\theta} + O\left(\hat{\theta}^2, \gamma \hat{\theta}, \gamma^2\right), \tag{5.9}$$

where $\psi^* = \psi(0, \theta_0)$ and $\eta^* = \eta(0, \theta_0) = -\frac{\partial \psi}{\partial \theta}\big|_{(0,\theta_0)}$. Similarly, ψ_1 for phase 2 can be written as

$$\psi_1 = \psi_1^* - \eta_1^* \hat{\theta}_1 + O\left(\hat{\theta}_1^2, \gamma_1 \hat{\theta}_1, \gamma_1^2\right), \tag{5.10}$$

where $\psi_1^* = \psi_1(0, \theta_0)$ and $\eta_1^* = \eta_1(0, \theta_0) = -\frac{\partial \psi_1}{\partial \theta_1}\big|_{(0,\theta_0)}$. We next assume that the heat flux q for phase 1 is zero when $\frac{d\theta}{dx} = 0$, and the heat flux q_1 for phase 2 is zero when $\frac{d\theta_1}{dx_1} = 0$. We additionally assume that $\left|\frac{d\theta}{dx}\right| \ll 1 \; \forall \, x \in [0, s(t))$ and $\left|\frac{d\theta_1}{dx_1}\right| \ll 1 \; \forall \, x_1 \in (s(t), L_1(t)]$, at each $t \in \Gamma$. For these assumptions, the first-order approximation of q for phase 1 is

$$q = k\frac{d\theta}{dx}, \tag{5.11}$$

where $k = \left[\partial \hat{q}/\partial\left(\frac{d\theta}{dx}\right)\right]_{(0,\theta_0,0)}$, and the first-order approximation of q_1 for phase 2 is

$$q_1 = k_1 \frac{d\theta_1}{dx_1}, \tag{5.12}$$

where $k_1 = \left[\partial \hat{q}_1/\partial\left(\frac{d\theta_1}{dx_1}\right)\right]_{(0,\theta_0,0)}$.[6] Using (5.9), (5.10), and (4.3)$_2$, the first-order approximation of the driving traction given by (4.5) is

$$f = \tilde{J}^* \psi_1^* - \psi^* - \left(\tilde{J}^* \eta_1^* - \eta^*\right)\hat{\theta}^{\pm} - \sigma^- \gamma_0, \tag{5.13}$$

where σ^- is given by the linearized form of (5.2)$_1$ and $\tilde{J}^* = 1 + \tilde{\gamma}(\theta_0) = 1 + \gamma_0$.[7] We must also specify the kinetic relation, which relates $\dot{s}$ and at least f. This will be discussed in the following section.

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[6] Note that Equations (5.11) and (5.12) both have the form of Fourier's Law.

[7] Note that because both phases are homogeneous, $\psi^-(0, \theta_0) = \psi^*$, $\eta^-(0, \theta_0) = \eta^*$, $\psi_1^+(0, \theta_0) = \psi_1^*$, and $\eta_1^+(0, \theta_0) = \eta_1^*$.

6. The Kinetic Relation

It is assumed that the thermoelastic material under consideration has a kinetic relation of the form

$$\dot{s} = \Phi(f, \theta), \tag{6.1}$$

where $\theta = \theta(s)$. Because of the entropy inequality $(4.4)_3$, this kinetic relation must be such that

$$f\Phi(f, \theta) \geq 0. \tag{6.2}$$

We additionally assume that at each temperature $\theta(s) \in \Omega$ the kinetic relation given by (6.1) has the form

$$\Phi(f, \theta) \begin{cases} > 0, & f > f_2(\theta) \\ = 0, & f_1(\theta) \leq f \leq f_2(\theta) \\ < 0, & f < f_1(\theta) \end{cases}, \tag{6.3}$$

where $f_1(\theta(s))$ and $f_2(\theta(s))$ depend on the material and are such that $f_1 \leq 0$ and $f_2 \geq 0 \ \forall \ \theta(s) \in \Omega$. We note that this kinetic relation satisfies (6.2). A kinetic relation that has the form given by (6.3) and is linear in f is

$$\Phi(f, \theta) = \begin{cases} \dfrac{1}{\nu_2(\theta)}\{f - f_2(\theta)\}, & f > f_2(\theta) \\ 0, & f_1(\theta) \leq f \leq f_2(\theta) \\ \dfrac{1}{\nu_1(\theta)}\{f - f_1(\theta)\}, & f < f_1(\theta) \end{cases}, \tag{6.4}$$

where $\nu_1(\theta(s)) \geq 0$ and $\nu_2(\theta(s)) \geq 0 \ \forall \ \theta(s) \in \Omega$. We note that because (6.4) is linear in f, it is probably most appropriate only for values of f such that $f_1 \leq f \leq f_2$, $|(f - f_2)/f| \ll 1$, or $|(f - f_1)/f| \ll 1$. For the

following, let $B(\theta_0) = \left\{ (\gamma^-, \gamma_1^+, \theta) / |\gamma^-| \ll 1, |\gamma_1^+| \ll 1, |\hat{\theta}| \ll 1 \right\}$ and $\Lambda(\theta_0) = \{ (f(\gamma^-, \gamma_1^+, \theta), \theta) / (\gamma^-, \gamma_1^+, \theta) \in B(\theta_0) \}$, for a given $\theta_0 \in \Omega$.[8] Three cases will be considered.

<u>Case I</u>: Consider first the case where the temperature $\theta_0 \in \Omega$ and the material are such that $f(0, 0, \theta_0) = f_1(\theta_0) = f_2(\theta_0) = 0$.[9] Additionally, assume that $(\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$ at each $t \in \Gamma$. For this case, $\left. \frac{df_1}{d\theta} \right|_{\theta_0} = 0$ and $\left. \frac{df_2}{d\theta} \right|_{\theta_0} = 0$, since $f_1(\theta_0) = f_2(\theta_0) = 0$ and it is required that $f_1 \leq 0$ and $f_2 \geq 0 \;\; \forall \, \theta(s) \in \Omega$. Thus, for this case, $f_1 = O(\hat{\theta}^2)$, $f_2 = O(\hat{\theta}^2)$, and the first-order approximation of (6.4) is

$$
\Phi = \begin{cases}
-\dfrac{1}{\nu_2} \left\{ \left(\tilde{J}^* \eta_1^* - \eta^* \right) \hat{\theta} + \sigma^- \gamma_0 \right\}, & -\left(\tilde{J}^* \eta_1^* - \eta^* \right) \hat{\theta} - \sigma^- \gamma_0 \geq 0 \\[3mm]
-\dfrac{1}{\nu_1} \left\{ \left(\tilde{J}^* \eta_1^* - \eta^* \right) \hat{\theta} + \sigma^- \gamma_0 \right\}, & -\left(\tilde{J}^* \eta_1^* - \eta^* \right) \hat{\theta} - \sigma^- \gamma_0 \leq 0
\end{cases},
$$
$$(6.5)$$

$\forall \, (\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$, where $\nu_1 = \nu_1(\theta_0) \geq 0$ and $\nu_2 = \nu_2(\theta_0) \geq 0$.

<u>Case II</u>: Consider the case where the temperature $\theta_0 \in \Omega$ and the material are such that $f(0, 0, \theta_0) = f_1(\theta_0)$ and $f_1(\theta_0)$ is not small, in the sense that $|f - f_1| < f_1$ $\forall \, (f, \theta) \in \Lambda(\theta_0)$. If phase 1 represents the austenite and phase 2 represents the martensite, the temperature θ_0 for this case might be considered to represent the martensitic start temperature of the material. We also assume that $(\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$ at each $t \in \Gamma$. For this case, the portion of (6.4) that corresponds to $(f, \theta) \in \Lambda(\theta_0)$ is

$$
\Phi(f, \theta) = \begin{cases}
0, & f \geq f_1(\theta) \\[3mm]
\dfrac{1}{\nu_1(\theta)} \{ f - f_1(\theta) \}, & f < f_1(\theta)
\end{cases}.
$$
$$(6.6)$$

Substituting (5.13) and

$$
f_1 = f_1(\theta_0) + \mu_1 \hat{\theta} + O(\hat{\theta}^2), \tag{6.7}
$$

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[8] Note that although f depends on $\theta(s)$, it can be varied independently of $\theta(s)$ since it also depends on γ^- and γ_1^+.
[9] Note that $f(0, 0, \theta_0) = \tilde{J}^* \psi_1^* - \psi^*$.

where $\mu_1 = \left.\frac{df_1}{d\theta}\right|_{\theta_0}$, into (6.6) and retaining only first-order terms yields

$$\Phi = \begin{cases} 0, & -\left(\tilde{J}^*\eta_1^* - \eta^*\right)\hat{\theta} - \sigma^-\gamma_0 \geq \mu_1\hat{\theta} \\ -\frac{1}{\nu_1}\left\{\left[\mu_1 + \left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]\hat{\theta} + \sigma^-\gamma_0\right\}, & -\left(\tilde{J}^*\eta_1^* - \eta^*\right)\hat{\theta} - \sigma^-\gamma_0 < \mu_1\hat{\theta} \end{cases} , \tag{6.8}$$

$\forall \, (\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$.

<u>Case III</u>: We next consider the case where the temperature $\theta_0 \in \Omega$ and the material are such that $f(0,0,\theta_0) = f_2(\theta_0)$ and $f_2(\theta_0)$ is not small, in the sense that $|f - f_2| < f_2$ $\forall \, (f,\theta) \in \Lambda(\theta_0)$. If phase 1 represents the austenite and phase 2 represents the martensite, the temperature θ_0 for this case might be considered to represent the austenite start temperature of the material. We also assume that $(\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$ at each $t \in \Gamma$. For this case, the portion of (6.4) that corresponds to $(f,\theta) \in \Lambda(\theta_0)$ is

$$\Phi(f,\theta) = \begin{cases} \frac{1}{\nu_2(\theta)}\{f - f_2(\theta)\}, & f > f_2(\theta) \\ 0, & f \leq f_2(\theta) \end{cases} . \tag{6.9}$$

Substituting (5.13) and

$$f_2 = f_2(\theta_0) + \mu_2\hat{\theta} + O(\hat{\theta}^2), \tag{6.10}$$

where $\mu_2 = \left.\frac{df_2}{d\theta}\right|_{\theta_0}$, into (6.9) and retaining only first-order terms yields

$$\Phi = \begin{cases} -\frac{1}{\nu_2}\left\{\left[\mu_2 + \left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]\hat{\theta} + \sigma^-\gamma_0\right\}, & -\left(\tilde{J}^*\eta_1^* - \eta^*\right)\hat{\theta} - \sigma^-\gamma_0 > \mu_2\hat{\theta} \\ 0, & -\left(\tilde{J}^*\eta_1^* - \eta^*\right)\hat{\theta} - \sigma^-\gamma_0 \leq \mu_2\hat{\theta} \end{cases} , \tag{6.11}$$

$\forall \, (\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$.[10]

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[10] Note that there are other cases concerning the linearized form of (6.4) that can be considered. For example, one can consider the case where $f_1(\theta_0)$ and/or $f_2(\theta_0)$ are small and nonzero.

Another method for obtaining a linear approximation of the nonzero portion of a kinetic relation of the form (6.1) is to simply expand it in a Taylor series about $(f, \theta) = (f_0, \theta_0)$, where $f_0 = f(0, 0, \theta_0)$. For the moment, assume that (6.1) and its first derivatives are continuous at $(f, \theta) = (f_0, \theta_0)$. For the case where $\Phi(f_0, \theta_0) = 0$, the Taylor series expansion of (6.1) about $(f, \theta) = (f_0, \theta_0)$ is

$$\Phi = \alpha_1(f - f_0) + \alpha_2\hat{\theta} + O\left((f - f_0)^2, (f - f_0)\hat{\theta}, \hat{\theta}^2\right), \qquad (6.12)$$

where $\alpha_1 = \frac{\partial \Phi}{\partial f}|_{(f_0, \theta_0)}$ and $\alpha_2 = \frac{\partial \Phi}{\partial \theta}|_{(f_0, \theta_0)}$.[11] Substituting (5.13) into (6.12) and retaining only first-order terms yields

$$\Phi = \left[\alpha_2 - \alpha_1\left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]\hat{\theta} - \alpha_1\sigma^-\gamma_0. \qquad (6.13)$$

For the case where α_1 is not continuous at f_0, α_1 is taken to be the limiting value of $\frac{\partial \Phi}{\partial f}$ as f approaches f_0 from the side of f corresponding to a nonzero portion of Φ. Comparing (6.5) with (6.13), we can conclude that $\alpha_1 = \frac{1}{\nu}$ and $\alpha_2 = 0$, where $\nu = \nu_1$ or $\nu = \nu_2$, depending on which branch of (6.5) is considered. Comparing (6.8) and (6.11) with (6.13), we can conclude that $\alpha_1 = \frac{1}{\nu}$ and $\alpha_2 = -\frac{\mu}{\nu}$, where $\nu = \nu_1$, $\mu = \mu_1$ or $\nu = \nu_2$, $\mu = \mu_2$, depending on the case considered.

7. The Solution of the Boundary Value Problem

From Equation $(4.2)_1$ and the boundary condition given by $(4.6)_2$, we can conclude that

$$\sigma_1(x_1) = \sigma_0 \qquad (7.1)$$

for $s(t) < x_1 \leq L_1(t)$, at each $t \in \Gamma$. From Equation $(4.1)_1$, the jump condition $(4.4)_1$, and (7.1) given above, we obtain

$$\sigma(x) = \sigma_0, \qquad (7.2)$$

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[11] Note that, although $f - f_0$ is not small for a relatively large range of values of $(\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$, $f - f_0$ does go to zero at the same rate as γ^-, γ_1^+, and $\hat{\theta}(s)$ go to zero.

for $0 \leq x < s(t)$, at each $t \in \Gamma$. Since the objective is to calculate the temperature at the interface and the corresponding kinetic relation, the displacements will not be calculated.

Using (5.11), Equation (4.1)$_2$ for phase 1 becomes

$$k\frac{d^2\hat{\theta}}{dx^2} = 0, \tag{7.3}$$

$\forall x \in (0, s(t))$, at each $t \in \Gamma$. Using (5.11) and (7.3) given above, Equation (4.1)$_3$ becomes

$$-k\left(\frac{1}{\theta}\frac{d\hat{\theta}}{dx}\right)^2 \leq 0, \tag{7.4}$$

$\forall x \in (0, s(t))$, at each $t \in \Gamma$. Additionally, the boundary condition (4.7)$_1$ in terms of $\hat{\theta}$ is

$$\hat{\theta}(0) = \bar{\theta}, \tag{7.5}$$

where $\bar{\theta} = \theta^* - \theta_0$. From (7.4), we obtain the well known result that $k \geq 0$. Assuming that $k > 0$, the solution of (7.3) that satisfies the boundary condition (7.5) is

$$\hat{\theta}(x) = C_1 x + \bar{\theta}, \tag{7.6}$$

for $0 \leq x < s(t)$, at each $t \in \Gamma$.

Using (5.12), Equation (4.2)$_2$ for phase 2 becomes

$$k_1\frac{d^2\hat{\theta}_1}{dx_1^2} = 0, \tag{7.7}$$

$\forall x_1 \in (s(t), L_1(t))$, at each $t \in \Gamma$. Using (5.12) and (7.7) given above, Equation

$(4.2)_3$ yields $k_1 \geq 0$. The boundary condition $(4.7)_2$ in terms of $\hat{\theta}_1$ is

$$\hat{\theta}_1(L_1(t)) = \bar{\theta}_1, \tag{7.8}$$

where $\bar{\theta}_1 = \theta_1^* - \theta_0$. Assuming that $k_1 > 0$, the solution of (7.7) that satisfies the boundary condition (7.8) is

$$\hat{\theta}_1(x_1) = C_3(x_1 - L_1(t)) + \bar{\theta}_1, \tag{7.9}$$

for $s(t) < x_1 \leq L_1(t)$, at each $t \in \Gamma$. From the continuity of temperature jump condition, we can solve for C_1 in (7.6) in terms of C_3 in (7.9) and obtain

$$\hat{\theta}(x) = \left\{ -C_3(1 + \gamma_0)(L - s) + \bar{\theta}_1 - \bar{\theta} \right\} \frac{x}{s} + \bar{\theta}, \tag{7.10}$$

for $0 \leq x \leq s(t)$, at each $t \in \Gamma$.

For the calculation of the first-order approximation of the energy jump condition, a kinetic relation which can have the forms of the kinetic relations given by (6.5), (6.8), and (6.11) will be used. In particular, we consider the kinetic relation which has the form

$$\dot{s} = \frac{1}{\nu(\theta)} \{ f - f^*(\theta) \}. \tag{7.11}$$

We next assume that the temperature θ_0 and the material are such that $f(0, 0, \theta_0) = f^*(\theta_0)$. Considering this and substituting (5.13) and

$$f^* = f^*(\theta_0) + \mu\hat{\theta} + O(\hat{\theta}^2), \tag{7.12}$$

where $\mu = \frac{df^*}{d\theta}\big|_{\theta_0}$, into (7.11) and retaining only first-order terms yields

$$\dot{s} = -\frac{1}{\nu} \left\{ \left[\mu + \left(\tilde{J}^* \eta_1^* - \eta^* \right) \right] \hat{\theta} + \sigma^- \gamma_0 \right\}, \tag{7.13}$$

$\forall\, (\gamma^-, \gamma_1^+, \theta) \in B(\theta_0)$, where $\sigma^- = \sigma_0$ for the problem considered here. This kinetic relation will be used for the boundary value problem under consideration. Using

(5.13), (7.13), (5.11), and (5.12), the first-order approximation of the energy jump condition given by $(4.4)_2$ is

$$-\frac{1}{\nu}\left\{\theta_0\left(\tilde{J}^*\eta_1^* - \eta^*\right) + f_0\right\}\left\{\left[\mu + \left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]\hat{\theta} + \sigma_0\gamma_0\right\}$$

$$+ \left(k_1\left(\frac{d\hat{\theta}_1}{dx_1}\right)^+ - k\left(\frac{d\hat{\theta}}{dx}\right)^-\right) = 0. \tag{7.14}$$

The constant C_3 in Equations (7.9) and (7.10) will be determined from this equation. We next define the nondimensional quantities

$$\tilde{s} = \frac{s}{L}, \quad \tilde{\theta}(\tilde{s}) = \frac{\hat{\theta}(\tilde{s}L)}{\theta_0}, \quad \tilde{\theta}^* = \frac{\bar{\theta}}{\theta_0}, \quad \tilde{\theta}_1^* = \frac{\bar{\theta}_1}{\theta_0}, \quad \tilde{k} = \frac{k_1}{\tilde{J}^*k}, \tag{7.15}$$

$$\tilde{\sigma}_0 = \frac{\sigma_0\gamma_0}{\theta_0\left[\mu + \left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]}, \quad \tilde{\nu} = \frac{\nu k}{L\left(\tilde{J}^*\varepsilon_1^* - \varepsilon^*\right)\left[\mu + \left(\tilde{J}^*\eta_1^* - \eta^*\right)\right]},$$

where ε_1^* is the internal energy of phase 2 evaluated at $(\gamma_1, \theta_1) = (0, \theta_0)$, ε^* is the internal energy of phase 1 evaluated at $(\gamma, \theta) = (0, \theta_0)$, and $\left(\tilde{J}^*\varepsilon_1^* - \varepsilon^*\right) = \left(\tilde{J}^*\psi_1^* - \psi^*\right) + \theta_0\left(\tilde{J}^*\eta_1^* - \eta^*\right)$. Substituting (7.9) and (7.10) into (7.14), using the nondimensional quantities given by (7.15), and then solving for C_3 yields

$$C_3 = \frac{\theta_0\left\{\tilde{\sigma}_0 + \tilde{\theta}_1^* + \dfrac{\tilde{\nu}(\tilde{\theta}_1^* - \tilde{\theta}^*)}{\tilde{s}}\right\}}{L(1 + \gamma_0)(1 - \tilde{s})\left\{1 + \dfrac{\tilde{\nu}}{\tilde{s}} + \dfrac{\tilde{\nu}\tilde{k}}{(1 - \tilde{s})}\right\}}. \tag{7.16}$$

Substituting (7.16) into (7.9) or (7.10) yields the following for the temperature at the interface:

$$\tilde{\theta}(\tilde{s}) = \frac{\left(\tilde{\nu}\,\tilde{\theta}^* - \tilde{\sigma}_0\tilde{s}\right)(1 - \tilde{s}) + \tilde{\nu}\,\tilde{k}\,\tilde{\theta}_1^*\tilde{s}}{(\tilde{\nu} + \tilde{s})(1 - \tilde{s}) + \tilde{\nu}\,\tilde{k}\tilde{s}}\,. \tag{7.17}$$

8. The Temperature at the Interface

The temperature at the interface given by (7.17) for the three cases described in Section 6 will be studied as the material coefficients $\tilde{\nu}$ and $\tilde{k}$, the temperature boundary conditions $\tilde{\theta}^*$ and $\tilde{\theta}_1^*$, the end load $\tilde{\sigma}_0$, and the position $\tilde{s}$ of the phase boundary are varied. The temperature at the interface for each of these three cases can be obtained from (7.17) by substituting the appropriate values of ν and μ. For Cases II and III, when f is in an interval such that $\dot{s} = 0$, the temperature at the interface can be obtained from (7.17) by letting $\nu \rightarrow \infty$.

Plots of $\tilde{\theta}(\tilde{s})$ vs $\tilde{s}$ for several values of $\tilde{\nu}$ are presented in Figures 1 and 2. In these figures, and in the rest of the figures in this section, the values of $\tilde{\theta}(\tilde{s})$, $\tilde{\theta}^*$, $\tilde{\theta}_1^*$, and $\tilde{\sigma}_0$ that are shown should all be multiplied by some small number; e.g. 10^{-5}. We note that such a multiplicative factor can be divided out in (7.17). The limiting values of $\tilde{\theta}(\tilde{s})$ as $\tilde{\nu} \rightarrow 0$ and $\tilde{\nu} \rightarrow \infty$ are

$$\lim_{\tilde{\nu}\to 0} \tilde{\theta}(\tilde{s}) = -\tilde{\sigma}_0\,,$$

$$\tag{8.1}$$

$$\lim_{\tilde{\nu}\to\infty} \tilde{\theta}(\tilde{s}) = \frac{\tilde{\theta}^*(1 - \tilde{s}) + \tilde{\theta}_1^*\tilde{k}\tilde{s}}{1 - \tilde{s} + \tilde{k}\tilde{s}}\,,$$

respectively. An easy calculation shows that as $\tilde{\nu}$ is varied $\frac{\partial \tilde{\theta}(\tilde{s})}{\partial \tilde{\nu}} \rightarrow 0$ only as $\tilde{\nu} \rightarrow \infty$, and that $\lim_{\tilde{\nu}\to\infty} \frac{\partial^2 \tilde{\theta}(\tilde{s})}{\partial \tilde{\nu}^2} = 0$. Therefore, as $\tilde{\nu}$ is varied, $\tilde{\theta}(\tilde{s})$ (at a given $\tilde{s}$) monotonically increases or decreases, depending on the values of the other

parameters.[12] Also, note that $(8.1)_2$ is the temperature at the interface for Cases II and III when f is in an interval such that $\dot{s} = 0$.

Plots of $\tilde{\theta}(\tilde{s})$ vs $\tilde{s}$ for several values of $\tilde{k}$ are presented in Figure 3. The limiting values of $\tilde{\theta}(\tilde{s})$ as $\tilde{k} \to 0$ and $\tilde{k} \to \infty$ are

$$\lim_{\tilde{k} \to 0} \tilde{\theta}(\tilde{s}) = \frac{-\tilde{\sigma}_0 \tilde{s} + \tilde{\nu}\,\tilde{\theta}^*}{\tilde{s} + \tilde{\nu}}\,,$$

(8.2)

$$\lim_{\tilde{k} \to \infty} \tilde{\theta}(\tilde{s}) = \tilde{\theta}_1^*\,,$$

respectively. An easy calculation shows that as $\tilde{k}$ is varied $\frac{\partial \tilde{\theta}(\tilde{s})}{\partial \tilde{k}} \to 0$ only as $\tilde{k} \to \infty$, and that $\lim_{\tilde{\nu} \to \infty} \frac{\partial^2 \tilde{\theta}(\tilde{s})}{\partial \tilde{k}^2} = 0$. Therefore, as $\tilde{k}$ is varied, $\tilde{\theta}(\tilde{s})$ monotonically increases or decreases, depending on the values of the other parameters.

Plots of $\tilde{\theta}(\tilde{s})$ vs $\tilde{s}$ for several values of $\tilde{\sigma}_0$ are presented in Figures 4 and 5. The limiting value of $\tilde{\theta}(\tilde{s})$ as $\tilde{\sigma}_0 \to 0$ can easily be observed from (7.17), and the limiting value of $\tilde{\theta}(\tilde{s})$ as $\tilde{\sigma}_0 \to \infty$ is ∞.

The values of $\tilde{\theta}(\tilde{s})$ as $\tilde{s} \to 0$ and $\tilde{s} \to 1$ are

$$\lim_{\tilde{s} \to 0} \tilde{\theta}(\tilde{s}) = \tilde{\theta}^*, \qquad \lim_{\tilde{s} \to 1} \tilde{\theta}(\tilde{s}) = \tilde{\theta}_1^*\,,$$

(8.3)

respectively. These results should be expected. An extremum of $\tilde{\theta}(\tilde{s})$ can, of course, be determined from

$$\frac{d\tilde{\theta}(\tilde{s})}{d\tilde{s}} = 0\,.$$

(8.4)

For a given set of material coefficients and boundary conditions, the values of $\tilde{s}$ that satisfy (8.4) are

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[12] Whether $\tilde{\theta}(\tilde{s})$ monotonically increases or decreases can, of course, be determined by comparing $(8.1)_1$ and $(8.1)_2$ for a given set of parameter values.

$$\tilde{s} = \frac{\tilde{\sigma}_0 + \tilde{\theta}^*}{\tilde{\sigma}_0\left(1 - \tilde{k}\right) + \tilde{\theta}^* - \tilde{k}\tilde{\theta}_1^*}$$

$$\pm \sqrt{\left(\frac{\tilde{\sigma}_0 + \tilde{\theta}^*}{\tilde{\sigma}_0\left(1 - \tilde{k}\right) + \tilde{\theta}^* - \tilde{k}\tilde{\theta}_1^*}\right)^2 + \frac{\tilde{\nu}\,\tilde{k}\left(\tilde{\theta}_1^* - \tilde{\theta}^*\right) - \left(\tilde{\sigma}_0 + \tilde{\theta}^*\right)}{\tilde{\sigma}_0\left(1 - \tilde{k}\right) + \tilde{\theta}^* - \tilde{k}\tilde{\theta}_1^*}} \qquad (8.5)$$

Note that for a given set of parameters, $\tilde{s}$ given by (8.5) is a physically meaningful solution only if it is real-valued and is such that $0 < \tilde{s} < 1$.

9. A Quasi-Static Process

In this section, a quasi-static process will be considered for the three cases described in Section 6. The direction of motion of the phase boundary will be considered and the static equilibrium positions of the phase boundary will be determined. For the following cases, let

$$\dot{\tilde{s}} = \frac{d\tilde{s}(\tilde{t})}{d\tilde{t}}, \qquad \tilde{t} = t\left(\frac{k\theta_0}{L^2\left(\tilde{J}^*\epsilon_1^* - \epsilon^*\right)}\right). \qquad (9.1)$$

Before we consider the three cases, however, we will first consider the signs of some of the variables that have the same signs for all of the cases considered. As is usually done, we assume that the dependence of the Gibbs free energy g^* for an unstressed phase 1 and the Gibbs free energy $\tilde{J}^*g_1^*$ for an unstressed phase 2 (with respect to R^+) on the temperature is qualitatively like that presented in Figure 6 (see [2]).[13] Because the Gibbs free energy of an unstressed material is equal to the Helmholtz free energy of that unstressed material, we have $\psi^* = g^*$ and $\psi_1^* = g_1^*$ for all temperatures

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[13] The Gibbs free energy of the phase 1 can be defined as $g = \psi - \sigma\gamma$ and the Gibbs free energy of phase 2 can be defined as $g_1 = \psi_1 - \sigma_1\gamma_1$.

where the undeformed configurations of both phases are unstressed. From this, we can con.lude that $\eta^* > \tilde{J}^*\eta_1^* \ \forall \ \theta \in \Omega$ for the case under consideration. We also always have $L > 0$, $\theta_0 \geq 0$, $\tilde{J} > 0$, $k > 0$, $k_1 > 0$, and $\nu \geq 0$. From this, we can conclude that $\tilde{k} \geq 0$. We can also conclude from this, and the kinetic relations presented in Section 6, that for a given σ_0, an increase in temperature will result in an increase in f, and a decrease in temperature will result in a decrease in f. Therefore, if phase 1 represents the austenite and phase 2 represents the martensite, which is what corresponds to Figure 6, an increase in temperature favors the austenite, and a decrease in temperature favors the martensite. This corresponds exactly with experimental observations [2], [5].

Case I: From the description of this case given in Section 6, we have $\tilde{J}^*\psi_1^* - \psi^* = 0$ and $\mu = 0$. We next make the additional assumption that $\nu_1 = \nu_2 = \nu$, and we let

$$\tilde{\nu}' = \tilde{\nu}|_{\mu=0}, \quad \tilde{\sigma}_0' = \tilde{\sigma}_0|_{\mu=0}. \tag{9.2}$$

For this case, the sign of $\dot{\tilde{s}}$ is opposite the sign of $\dot{s}$, the sign of $\tilde{\sigma}_0'$ is opposite the sign of $\sigma_0\gamma_0$, $\tilde{\nu}' \geq 0$, and the kinetic relation given by (6.5) in terms of the nondimensional variables is

$$\dot{\tilde{s}} = -\frac{1}{\tilde{\nu}'}\left(\tilde{\theta}' + \tilde{\sigma}_0'\right), \tag{9.3}$$

where $\tilde{\theta}'$ is the temperature at the interface given by (7.17) with $(\tilde{\nu}, \tilde{\sigma}_0) = (\tilde{\nu}', \tilde{\sigma}_0')$. For the quasi-static process under consideration, static equilibrium occurs when $\dot{\tilde{s}} = 0$. From (9.3), we can conclude that this occurs when

$$\tilde{\theta}(\tilde{s}) = -\tilde{\sigma}_0'. \tag{9.4}^{14}$$

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14 In the remainder of this section, the temperature at the interface may sometimes be referred to as simply $\tilde{\theta}(\tilde{s})$, with the understanding that it corresponds to the case under consideration.

Using (7.17) in (9.4), the static equilibrium position of the phase boundary for a given set of material coefficients and boundary conditions will occur at

$$\tilde{s} = \frac{\tilde{\theta}^* + \tilde{\sigma}_0'}{\tilde{\theta}^* - k\tilde{\theta}_1^* + \left(1 - \tilde{k}\right)\tilde{\sigma}_0'} \,, \tag{9.5}$$

with the requirement that $0 < \tilde{s} < 1$. We note that in Figures 1 and 2, where plots of $\tilde{\theta}(\tilde{s})$ vs $\tilde{s}$ for several values of $\tilde{\nu}$ are presented, the static equilibrium positions given by (9.5) occur where all of the temperature curves intersect. This is because the static equilibrium position of the phase boundary given by (9.5) is independent of $\tilde{\nu}$. From (9.3) and the fact that the sign of $\dot{\tilde{s}}$ is opposite the sign of $\dot{s}$, we can conclude that $\dot{s} > 0$ for $\tilde{\theta}(\tilde{s}) > -\tilde{\sigma}_0'$, and $\dot{s} < 0$ for $\tilde{\theta}(\tilde{s}) < -\tilde{\sigma}_0'$. Because of this, the static equilibrium positions in Figures 1 and 4 are unstable, in the sense that the phase boundary will move away from these static equilibrium positions for any perturbation away from them, and the static equilibrium positions in Figures 2, 3, and 5 are stable. In Figures 4 and 5, some values of $\tilde{\sigma}_0$ are used which result in the static equilibrium position determined by (9.5) being outside of the interval $(0, 1)$. Consequently, the static equilibrium states of the bar for these values of $\tilde{\sigma}_0$ consist of only one phase.

The case where $\tilde{J}^*\eta_1^* = \eta^*$ in addition to $\tilde{J}^*\psi_1^* = \psi^*$ can also be considered here.[15] In particular, for this case, the first-order approximation of the energy jump condition reduces to the continuity of heat fluxes across the phase boundary. From (7.14), we can conclude that the temperature solution for this case is the same as the temperature solution for the case where $\nu \to \infty$. Thus, the temperature and quasi-static motions for the case where $\tilde{J}^*\psi_1^* = \psi^*$ and $\tilde{J}^*\eta_1^* = \eta^*$ are the same as they are for the cases presented here for Case I where $\tilde{\nu} \to \infty$.

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[15] These assumption might be most appropriate for the case where phase 1 and phase 2 represent two different variants of the same martensite.

<u>Case II</u>: For this case, let

$$\tilde{\nu}_1 = \tilde{\nu}\big|_{(\nu,\mu)=(\nu_1,\mu_1)}, \qquad \tilde{\sigma}_1 = \tilde{\sigma}_0\big|_{\mu=\mu_1}. \tag{9.6}$$

From the description of this case given in Section 6, we have $\tilde{J}\psi_1^* < \psi^*$. If we also make the assumption that $|\mu| < |J^*\eta_1^* - \eta^*|$, the sign of $\dot{\tilde{s}}$ will be opposite that of $\dot{s}$, the sign of $\tilde{\sigma}_1$ will be opposite that of $\sigma_0\gamma_0$, and $\tilde{\nu}_1 \geq 0$. The kinetic relation given by (6.8) in terms of the nondimensional variables is

$$\dot{\tilde{s}} = \begin{cases} 0, & \left(\tilde{\theta}^{(2)} + \tilde{\sigma}_1\right) \geq 0 \\[2mm] -\dfrac{1}{\tilde{\nu}_1}\left(\tilde{\theta}^{(1)} + \tilde{\sigma}_1\right), & \left(\tilde{\theta}^{(1)} + \tilde{\sigma}_1\right) \leq 0 \end{cases}. \tag{9.7}$$

where $\tilde{\theta}^{(1)}$ is the temperature at the interface given by (7.17) with $(\tilde{\nu}, \tilde{\sigma}_0) = (\tilde{\nu}_1, \tilde{\sigma}_1)$, and $\tilde{\theta}^{(2)}$ is given by $(8.1)_2$, which corresponds to (7.17) with $\nu \to \infty$. In Figures 7 and 8, plots of $\tilde{\theta}(\tilde{s})$ vs $\tilde{s}$ for $\tilde{\nu} = 0.1$ and $\tilde{\nu} \to \infty$ are presented. The case where $\tilde{\nu} = 0.1$ represents $\tilde{\theta}^{(1)}$ with $\tilde{\nu}_1 = 0.1$, and the case where $\nu \to \infty$ represents $\tilde{\theta}^{(2)}$. Quasi-static processes will be considered for the cases presented in these two figures. The position of the phase boundary corresponding to $\tilde{\theta}(\tilde{s}) = -\tilde{\sigma}_0$ is given by (9.5) with $\tilde{\sigma}_0' = \tilde{\sigma}_1$. This static equilibrium position also corresponds to the point where $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ intersect, since it is independent of $\tilde{\nu}$. When $\tilde{\theta}\big(\tilde{s}(\tilde{t}_0)\big) < -\tilde{\sigma}_1$, the temperature at the interface is given by $\tilde{\theta}^{(1)}$, and when $\tilde{\theta}\big(\tilde{s}(\tilde{t}_0)\big) > -\tilde{\sigma}_1$, the temperature at the interface is given by $\tilde{\theta}^{(2)}$. From (9.7), we can conclude that a phase boundary with an initial position of $\tilde{s}(\tilde{t}_0)$ such that $\tilde{\theta}\big(\tilde{s}(\tilde{t}_0)\big) > -\tilde{\sigma}_1$ will stay at that position, and a phase transformation will not occur. However, a phase boundary with an initial position such that $\tilde{\theta}\big(\tilde{s}(\tilde{t}_0)\big) < -\tilde{\sigma}_1$ will have a $\dot{s} < 0$, and consequently phase 1 will be converted into phase 2.

<u>Case III</u>: For this case, let

$$\tilde{\nu}_2 = \tilde{\nu}\big|_{(\nu,\mu)=(\nu_2,\mu_2)}, \qquad \tilde{\sigma}_2 = \tilde{\sigma}_0\big|_{\mu=\mu_2}. \qquad (9.8)$$

From the description of this case given in Section 6, we have $\psi^* < \tilde{J}\psi_1^*$. Also, for the following, we assume that $|\mu| < |J^*\eta_1^* - \eta^*|$. For this case, the sign of $\tilde{\nu}$ and the sign of $\dot{s}(t)$ in relation to $\dot{\bar{s}}$ depend on the relative values of $(J\psi_1^* - \psi^*)$ and $\theta_0(J\eta_1^* - \eta^*)$. Thus, we cannot make such definite conclusions concerning the sign of $\tilde{\nu}$ and the sign of $\dot{s}(t)$ in relation to $\dot{\bar{s}}$ for this case, as we were able to do for the two cases discussed previously. The kinetic relation given by (6.11) in terms of the nondimensional variables is

$$\dot{\bar{s}} = \begin{cases} -\dfrac{1}{\tilde{\nu}_2}\left(\tilde{\theta}^{(2)} + \tilde{\sigma}_2\right), & \left(\tilde{\theta}^{(2)} + \tilde{\sigma}_2\right) \geq 0 \\[2mm] 0, & \left(\tilde{\theta}^{(1)} + \tilde{\sigma}_2\right) \leq 0 \end{cases}. \qquad (9.9)$$

where $\tilde{\theta}^{(1)}$ is the temperature at the interface given by $(8.1)_2$, which corresponds to (7.17) with $\nu \to \infty$, and $\tilde{\theta}^{(2)}$ is the temperature at the interface given by (7.17) with $(\tilde{\nu}, \tilde{\sigma}_0) = (\tilde{\nu}_2, \tilde{\sigma}_2)$. For this case, we will consider a quasi-static process for the cases presented in Figures 7 and 8. The position of the phase boundary corresponding to $\tilde{\theta}(\bar{s}) = -\tilde{\sigma}_2$ is given by (9.5) with $\tilde{\sigma}_0' = \tilde{\sigma}_2$. This static equilibrium position also corresponds to the point where $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ intersect, since it is independent of $\tilde{\nu}$. When $\tilde{\theta}\big(\bar{s}(\tilde{t}_0)\big) < -\tilde{\sigma}_1$, the temperature at the interface is given by $\tilde{\theta}^{(1)}$, and when $\tilde{\theta}\big(\bar{s}(\tilde{t}_0)\big) > -\tilde{\sigma}_1$, the temperature at the interface is given by $\tilde{\theta}^{(2)}$. From (9.9), we can conclude that a phase boundary with an initial position of $\bar{s}(\tilde{t}_0)$ such that $\tilde{\theta}\big(\bar{s}(\tilde{t}_0)\big) < -\tilde{\sigma}_2$ will stay at that position, and a phase transformation will not occur. However, a phase boundary with an initial position such that $\tilde{\theta}\big(\bar{s}(\tilde{t}_0)\big) > -\tilde{\sigma}_2$ will have a $\dot{s} > 0$, and consequently phase 2 will be converted into phase 1.

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